Fundamentals of Mathematics
Spring 2007, Proof 20
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Show directly from the definition of a metric that the function $d : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d(x, y) = |x - y|, \forall (x, y) \in \mathbb{R}^2$$

is a metric on $\mathbb{R}$.

Proof. It suffices to show that the function $d(x, y)$ satisfies the following axioms:

(i) $(\forall x, y \in \mathbb{R})\ d(x, y) \geq 0$ with equality if and only if $x = y$,

(ii) $(\forall x, y \in \mathbb{R})\ d(x, y) = d(y, x)$,

(iii) $(\forall x, y, z \in \mathbb{R})\ d(x, y) \leq d(x, z) + d(z, y)$.

First, we argue (i). By definition, for all $z \in \mathcal{F}$, if $z \geq 0$, then $|z| = z \geq 0$. Otherwise, z must be less than 0, but still $|z| = -z \geq 0$. Thus, $|z| \geq 0$, $\forall z \in \mathcal{F}$. It follows that $d(x, y) = |x - y| \geq 0$. Furthermore, it is clear that $|x - y| = 0$ if and only if $x - y = 0$, or, $x = y$.

Second, we argue (ii). It has been shown that $|-z| = |z|$, $\forall z \in \mathcal{F}$.[1] Thus, we may write,

$$d(x, y) = |x - y| = |-(-x + y)| = |-x + y| = |y - x| = d(y, x).$$

Third, we argue (iii). It suffices to show that $|x - y| \leq |x - z| + |z - y|$. Using the triangle inequality, we may write,

$$|x - y| = |x - z + z - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y|.$$

This completes the proof.

[1] Refer to workbook problem 4.2.9.