Fundamentals of Mathematics
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Andrew Cheong

Let V denote an inner product space (over $\mathbb{R}$). Define $d : V \times V \to \mathbb{R}$ by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||, \forall \mathbf{u}, \mathbf{v} \in V$$

where the indicated norm is the norm induced by the inner product. Prove that d is a metric on V.

Proof. It suffices to show that the function $d(\mathbf{u}, \mathbf{v})$ satisfies the following axioms:

(i) $(\forall \mathbf{u}, \mathbf{v} \in V)\ d(\mathbf{u}, \mathbf{v}) \geq 0$ with equality if and only if $u = v$,

(ii) $(\forall \mathbf{u}, \mathbf{v} \in V)\ d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$,

(iii) $(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V)\ d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$.

First, we argue (i). The axioms of positivity and definiteness of a norm verify this statement.

Second, we argue (ii). It suffices to show that $||\mathbf{u} - \mathbf{v}|| = ||\mathbf{v} - \mathbf{u}||$.

$$||\mathbf{u} - \mathbf{v}|| = ||(-1)(\mathbf{v} - \mathbf{u})|| = |-1|||\mathbf{v} - \mathbf{u}|| = ||\mathbf{v} - \mathbf{u}||$$

Third, we argue (iii). It suffices to show that $||\mathbf{u} - \mathbf{v}|| \leq ||\mathbf{u} - \mathbf{w}|| + ||\mathbf{w} - \mathbf{v}||$. Using the triangle inequality, we may write,

$$||\mathbf{u} - \mathbf{v}|| = ||\mathbf{u} - \mathbf{w} + \mathbf{w} - \mathbf{v}|| = ||(\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v})|| \leq ||\mathbf{u} - \mathbf{w}|| + ||\mathbf{w} - \mathbf{v}||.$$

This completes the proof.