Fundamentals of Mathematics
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Problem. Let S be a subset of a metric space, M. Prove that S is closed if and only if the limit of every sequence in S (that converges in M) belongs to S.

Proof. It suffices to show that (i) if S is closed, then the limit of every sequence in S (that converges in M) belongs to S; and that (ii) if the limit of every sequence in S (that converges in M) belongs to S, then S is closed.

First, we argue (i). Let $\{x_n\}_{n=1}^{\infty}$ denote an infinite sequence in S that converges to an element $x \in M$. It suffices to show, given that S is closed, that $x \in S$.

Suppose $x \in S^c$. Since S is closed, S^c must be open. Since S^c is open, by definition, there exists a $B(x, r^*) \subseteq S^c$ for some $r^* > 0$. Note that since $\{x_n\}_{n=1}^{\infty}$ is an infinite sequence in S, no element x_n belongs to S^c. It follows that no element x_n belongs to any subset of S^c, namely $B(x, r^*)$, i.e.

$$x_n \notin B(x, r^*), \forall n \in \mathbb{Z}^+.$$

Since $\{x_n\}_{n=1}^{\infty}$ converges to x, by definition, for every $\epsilon > 0$, there exists an $N \in \mathbb{Z}^+$ such that $d(x, x_n) < \epsilon$ for all $n > N$. **Or, equivalently, for every $r > 0$, there exists an $N \in \mathbb{Z}^+$ such that $x_n \in B(x, r)$ for all $n > N$.** However, under the supposition that $x \in S^c$, we have found a $B(x, r^*)$ in which no element x_n exists. Thus, it cannot be true that $x \in S^c$; we conclude that $x \in S$.

This completes the proof of (i).

Next, we argue (ii). Let $\{x_n\}_{n=1}^{\infty}$ denote an infinite sequence in S that converges to x. Suppose S is open. Then, since for all x_n, $n \in \mathbb{Z}^+$, there exists an $r > 0$ such that $B(x_n, r) \in S$, one is able to define an element $x \notin S$ such that $d(x, x_n) < \epsilon$ for all $n > N$. It follows that one is able to define an infinite sequence in S that converges to $x \notin S$. Thus, if S is open, then there exists an infinite sequence in S that converges to $x \notin S$. The contrapositive statement is: if all infinite sequences in S converge to $x \in S$, then S is closed. This completes the proof of (ii).

This completes the proof.