Fundamentals of Mathematics
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Let S denote a non-empty subset of a metric space M. Let ∂S denote the set of boundary points of S. Prove that ∂S is a closed subset of M.

Proof. It suffices to show that ∂S^c is open. In other words, it suffices to show that for every $x \in \partial S^c$, there exists an $r > 0$ such that $B(x, r) \subseteq \partial S^c$.

Let x be an arbitrary element in ∂S^c. Since x is not a boundary point, there exists an $r > 0$ such that either $B(x, r) \subseteq S$ or $B(x, r) \subseteq S^c$. We argue that in either case, $B(x, r)$ is wholly contained in ∂S^c, i.e. no element in $B(x, r)$ is in ∂S.

> Let y be an arbitrary element in $B(x, r)$. Since $B(x, r)$ is open, by definition, there exists an $r' > 0$ such that $B(y, r') \subseteq B(x, r)$. Since either $B(x, r) \cap S = \emptyset$ or $B(x, r) \cap S^c = \emptyset$, it follows that either $B(y, r') \cap S = \emptyset$ or $B(y, r') \cap S = \emptyset$. Therefore, in either case, $y \notin \partial S$.

Since for an arbitrary $x \in \partial S^c$, there exists an $r > 0$ such that $B(x, r) \subseteq \partial S^c$, ∂S^c is open, and ∂S is closed.

This completes the proof.