Fundamentals of Mathematics
Spring 2007, Proof 7
Andrew Cheong

Prove that the product of two integers is an integer.

Proof. It suffices to prove the following statements:

i. The product of two integers, at least one of which is positive, is an integer.

ii. The product of two integers, at least one of which is zero, is an integer.

iii. The product of two integers, at least one of which is negative, is an integer.

First, we argue (i).

Let $S = \{m \in \mathbb{Z} :$ (a) $m \geq 1$, and (b) $mk \in \mathbb{Z}, \forall k \in \mathbb{Z}\}$. We argue, by induction on m, that $S = \{m \in \mathbb{Z} : m \geq 1\}$.

First, we argue the basis step by showing that $1 \in S$. For the case of $m = 1$, condition (a) is satisfied since $1 \geq 1$. Condition (b) is satisfied since $1 \cdot k = k \in \mathbb{Z}, \forall k \in \mathbb{Z}$. Thus, $1 \in S$.

Next, we argue the induction step by assuming that $n \in S$ and showing that $n+1 \in S$. Since $n \in S$, it follows that (a) $n \geq 1$ and (b) $nk \in \mathbb{Z}, \forall k \in \mathbb{Z}$. For the case of $m = n+1$, condition (a) is satisfied since $n+1 > n \geq 1$. Condition (b) is satisfied because

$$(n+1)k = nk + 1 \cdot k = nk + k \in \mathbb{Z}.$$ [1]

Thus, $n+1 \in S$.

This completes the proof of statement (i).

Next, we argue (ii).

Let $x \in \mathbb{Z}$. $0x = 0 \in \mathbb{Z}$. This completes the proof of statement (ii).

[1] That the sum of two integers is an integer is shown in the course notes.

Finally, we argue (iii) in a manner similar to that of (i).

Let $S = \{m \in \mathbb{Z} :$ (a) $m \geq 1$, and (b) $-mk \in \mathbb{Z}, \forall k \in \mathbb{Z}\}$. We argue, by induction on m, that $S = \{m \in \mathbb{Z} : m \geq 1\}$.

First, we argue the basis step by showing that $1 \in S$. For the case of $m = 1$, condition (a) is satisfied since $1 \geq 1$. Condition (b) is satisfied since $-1 \cdot k = -k \in \mathbb{Z}, \forall k \in \mathbb{Z}$.[2] Thus, $1 \in S$.

Next, we argue the induction step by assuming that $n \in S$ and showing that $n+1 \in S$. Since $n \in S$, it follows that (a) $n \geq 1$ and (b) $-nk \in \mathbb{Z}, \forall k \in \mathbb{Z}$. For the case of $m = n+1$, condition (a) is satisfied since $n+1 > n \geq 1$. Condition (b) is satisfied because

$$-(n+1)k = (-n + (-1))k = -nk + (-1)k = -nk + (-k) \in \mathbb{Z}.$$

Thus, $n+1 \in S$.

This completes the proof of statement (iii).

This completes the proof.

[2] That $-k \in \mathbb{Z}, \forall k \in \mathbb{Z}$ is shown in workbook problem 4.4.7.