Fundamentals of Mathematics
Spring 2007, Proof 8
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Prove that an infinite sequence in $\mathbb{R}$ can have at most one limit in $\mathbb{R}$.

Proof. Let $\lim_{n\to\infty} x_n = x_1$ and $\lim_{n\to\infty} x_n = x_2$. It suffices to show that $x_1 = x_2$.

By definition, it is given that for all $\epsilon > 0$, there exists $N_1 \in \mathbb{Z}$ such that $|x_n - x_1| < \epsilon, \forall n > N_1$. Also by definition, it is given that for all $\epsilon > 0$, there exists $N_2 \in \mathbb{Z}$ such that $|x_n - x_2| < \epsilon, \forall n > N_2$. Letting $N = max\{N_1, N_2\}$, the sum of the inequalities yield

$$|x_n - x_1| + |x_n - x_2| \leq 2\epsilon \qquad \forall n > N$$

Using the triangle inequality, it follows that

$$\begin{aligned} |x_n - x_1 + x_n - x_2| &\leq 2\epsilon && \forall n > N \\ |2x_n - (x_1 + x_2)| &\leq 2\epsilon && \forall n > N \\ |x_n - \left(\frac{x_1}{2} + \frac{x_2}{2}\right)| &\leq \epsilon && \forall n > N \end{aligned}$$

By definition, this is equivalent to $\lim_{n\to\infty} x_n = \frac{x_1}{2} + \frac{x_2}{2}$. Since $\lim_{n\to\infty} x_n = x_1$,

$$\begin{aligned} x_1 &= \frac{x_1}{2} + \frac{x_2}{2} \\ \frac{x_1}{2} &= \frac{x_2}{2} \\ x_1 &= x_2 \end{aligned}$$

This completes the proof.