Fundamentals of Mathematics
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Let D denote a subset of $\mathbb{R}$ and let $x_0 \in D$. Let f and g be real-valued functions defined on D, both continuous at x_0. Prove that $f\,g$ is continuous at x_0.

Proof. Let $h(x) = f(x) + g(x)$. Since $f(x)$ and $g(x)$ are continuous, and the sum of two continuous functions is continuous, it follows that $h(x)$ is continuous. Next, the following computation shows that $f(x)g(x)$ is continuous as well.

$$\begin{aligned} h^2(x) &= [f(x) + g(x)]^2 \\ h^2(x) &= f^2(x) + g^2(x) + 2f(x)g(x) \\ f(x)g(x) &= \frac{1}{2}h^2(x) - \frac{1}{2}f^2(x) - \frac{1}{2}g^2(x) \end{aligned}$$

Since $f(x)$, $g(x)$, and $h(x)$ are continuous, squares of continuous functions are continuous, scalar multiples of continuous functions are continuous, and the sum of continuous functions is continuous, it follows that $f(x)g(x)$ is continuous.

This completes the proof.